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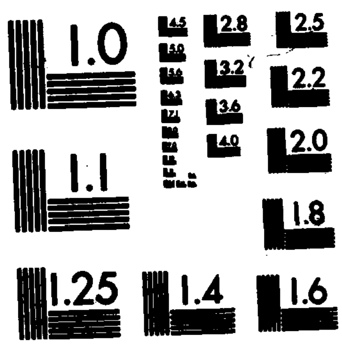

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Resonant Frequencies and Damping of a Liquid Drop With a Rigid Spherical Core

by C. A. Morrison

Abstract

Rayleigh's method is used to obtain the resonant frequencies of a liquid drop with a rigid core. The decay time of the drop is calculated by using Rayleigh's dissipation function and by assuming that the fluid in the drop and the external medium are viscous. It is shown that special cases reduce to known results.

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1. INTRODUCTION

In the formation of small hail within the atmosphere, conditions exist in which the inner core is ice, with a layer of water on the outside. Such a composite drop has resonant frequencies, and if both the air surrounding the drop and the liquid of the drop are viscous, these resonances are damped. Such resonances and perhaps the damping may be observed by techniques used to observe oscillations in completely liquid drops.[1-3]

In this report we investigate the resonant frequencies and damping for a composite drop. Both the liquid of the drop and the external medium (air) will be assumed to be viscous. The technique used is that of Rayleigh,[4] which we have used previously for determining the resonant frequencies of a liquid drop in an electric field.[5] The damping will be treated in a manner similar to that used in Morrison et al.[5]

2. KINETIC AND POTENTIAL ENERGY OF THE DROP

The inner rigid core of radius b is surrounded by a fluid of density ρ_i and viscosity η_i. The exterior of the drop of quiescent radius a is surrounded by fluid or air of density ρ_o and viscosity η_o. As in our previous analysis,[5] we assume that the outer surface of the drop is given by*

$$r(\theta,t) = a_o(t) + \sum_n{}' \, a_n(t)\, P_n(\cos a) \quad , \qquad (2.1I)$$

*Equation numbers containing an I are found by that number in reference (5). Thus, (2.1I) is (2.1) in reference (5).

where the prime on the sum indicates that the term n = 0 is missing. The constraint that the volume of the liquid contained in the outer shell is constant is the same as in Morrison et al.[5] To show this, we see that the volume of the liquid in the drop is

$$V = 2\pi \int_{-1}^{1} \left(\frac{r^3}{3} - \frac{b^3}{3}\right) d\mu \quad , \tag{1}$$

where $d\mu = -d(\cos\theta)$, and the equilibrium volume is given by

$$V = \frac{4\pi}{3} (a^3 - b^3) \quad . \tag{2}$$

If the result in equation (1) is equated to that of equation (2), the volume of the inner sphere cancels, and the constraint becomes identical to that of a liquid sphere, giving

$$a_o = a\left(1 - \frac{1}{a^2} \sum_n{}' \frac{a_n^2}{2n + 1}\right) \quad . \tag{2.2I}$$

The energy due to surface interface tension, γ, at the outer surface is obviously the same as in Morrison et al,[5] or

$$U_s = 4\pi\gamma \left[a^2 + \sum_n{}' \frac{(n - 1)(n + 2)}{2(2n + 1)} a_n^2\right] \quad . \tag{2.3I}$$

We shall assume that the velocity of the fluid in both regions is derivable from a velocity potential, so that

$$v_i = -\nabla\phi_i \qquad b < r < r(\theta,t)$$

$$v_o = -\nabla\phi_o \qquad r(\theta,t) < r \quad , \tag{3}$$

with ϕ_i and ϕ_o given by suitable solutions to Laplace's equation. The total kinetic energy is then given by

$$T = \frac{1}{2}\rho_i \int (\nabla\phi_i)^2 \, d\tau + \frac{1}{2}\rho_o \int (\nabla\phi_o)^2 d\tau \quad , \tag{4}$$

where the volume integrals cover the regions given in equation (3). The volume integrals in equation (4) can be converted to surface integrals by

$$\int (\nabla\phi)^2 d\tau = \int \phi(\nabla\phi) \cdot d\sigma \quad , \tag{5}$$

where $d\sigma$ is the incremental surface area directed outward from the volume. Using this result in the first integral in equation (4), we get

$$\int (\nabla\phi_i)^2 \, d\tau = 2\pi \int_{-1}^{1} d\mu \, r^2(\theta,t) \, \phi_i \left.\frac{\partial\phi_i}{\partial r}\right|_{r=r(\theta,t)} d\mu \quad , \tag{6}$$

where all the functions in the integrand are evaluated at $r(\theta,t)$ and where we have used the condition $v_r = 0$ at $r = b$. The second integral in equation (4) can be evaluated in the same manner to give

$$\int (\nabla\phi_o)^2 \, d\tau = -2\pi \int_{-1}^{1} d\mu \, r^2(\theta,t) \, \phi_o \left.\frac{\partial\phi_o}{\partial r}\right|_{r=r(\theta,t)} , \qquad (7)$$

where we have assumed that $v_r = 0$ at $r \to \infty$.

The results given in equations (6) and (7) can be used to obtain the kinetic energy in equation (4) once the velocity potentials are determined. The appropriate solutions of Laplace's equation for the two regions are

$$\phi_i = \sum_n \left(\frac{A_n}{r^{n+1}} + r^n B_n\right) P_n(\cos\theta) \quad , \; b < r < r(\theta,t) \quad ,$$

$$\phi_o = \sum_n \frac{C_n}{r^{n+1}} P_n(\cos\theta) \quad , \; r(\theta,t) < r < \infty \quad , \qquad (8)$$

where A_n, B_n, and C_n are constants to be determined by the boundary conditions. Also, in selecting the appropriate solution for the exterior region, $(r > r(\theta,t))$, we have chosen the solution that vanishes at infinity because we assume that the fluid is at rest far from the drop.

Before obtaining explicit expressions for the constants in equation (8), it is convenient to evaluate the terms in the kinetic energy by using these expressions. For the exterior region, we use the second equation (8) in equation (7) to obtain

$$\int (\nabla\phi_o)^2\, d\tau = -4\pi \sum_n \frac{(n+1)}{(2n+1)} \frac{C_n^2}{a^{2n+1}} \quad , \tag{9}$$

where we have taken r = a in the integrand of equation (7); higher order terms give corrections of the order a_n^3, which we are ignoring. Using equation (8) in equation (6), we get for the interior region

$$\int (\nabla\phi_i)^2\, d\tau = 4\pi \sum_n \frac{a^{2n+1}}{2n+1} \left[nB_n - \frac{(n+1)A_n}{a^{2n+1}}\right] \left(B_n + \frac{A_n}{a^{2n+1}}\right) \quad , \tag{10}$$

where, as in equation (9) we have taken $r(\theta,t) = a$ for the same reason as given in the derivation of equation (9). The results given in equations (9) and (10) can be used to calculate the kinetic energy once the constants are evaluated.

Since we are assuming the inner sphere to be rigid, we have $v_r = 0$ at r = b, and by using equations (3) and (8) we obtain

$$A_n = \frac{nb^{2n+1}}{n+1} B_n \quad . \tag{11}$$

At the boundary $r = r(\theta,t)$, we have the condition that $v_r = \dot{r}(\theta,t)$. Then from equations (3), (8), and (11), we get

$$B_n = \frac{-\dot{a}_n}{na^{n-1}\Gamma_n} \quad , \tag{12}$$

where

$$\Gamma_n = 1 - \left(\frac{b}{a}\right)^{2n+1} \quad .$$

For the external region, we have from equations (3) and (8) the result

$$C_n = \frac{a^{n+2}}{(n+1)}\dot{a}_n \quad . \tag{13}$$

The results given in equations (11) and (12), when substituted into equations (10), give

$$\int (\nabla\phi_i)^2 d\tau = 4\pi a^3 \sum_n \frac{(2n + 1 - n\Gamma_n)\dot{a}_n^2}{n(n+1)(2n+1)\Gamma_n} \quad . \tag{14}$$

Similarly, when the result given in equation (13) is substituted into equation (9), we get

$$\int (\nabla \phi_o)^2 \, d\tau = 4\pi a^3 \sum_n \frac{\dot{a}_n^2}{(n + 1)(2n + 1)} \quad . \tag{15}$$

Finally, the kinetic energy is given by using equations (14) and (15) in equation (4), or

$$T = 2\pi a^3 \sum_n \left[\frac{\rho_i\left(2n + 1 - n\Gamma_n\right)}{n(n + 1)(2n + 1)\Gamma_n} + \frac{\rho_o}{(n + 1)(2n + 1)}\right]\dot{a}_n^2 \quad . \tag{16}$$

In the absence of losses, the results given in equation (16) and in equation (2.3I) can be used to obtain the equation of motion for the a_n [$L = T - U_s$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{a}_n} - \frac{\partial L}{\partial a_n} = 0$]; see equation (2.8I).

$$a^3\left[\frac{\rho_i\left(2n + 1 - n\Gamma_n\right)}{n\Gamma_n} + \rho_o\right] \frac{\ddot{a}_n}{(n + 1)} + \gamma(n - 1)(n + 2)a_n = 0 \quad . \tag{17}$$

The resonant frequency, ω_n, for a_n is then given by

$$\omega_n^2 = \frac{\gamma(n - 1)(n + 1)(n + 2)n\Gamma_n}{a^3\left[\rho_i(2n + 1 - n\Gamma_n) + \rho_o n\Gamma_n\right]} \quad . \tag{18}$$

Several checks on the validity of equation (18) can now be made. If in equation (18) we let $b = 0 (\Gamma_n = 1)$ and $\rho_o = 0$, we obtain Rayleigh's result,[4]

$$\omega_n^2 = \frac{\gamma(n - 1)n(n + 2)}{\rho_i a^3} \quad . \tag{19}$$

Of course, if $b = a(\Gamma_n = 0)$ and there is no fluid to oscillate, consequently $\omega_n = 0$ results. If we let $b = 0$ in equation (18), then we obtain the resonant frequency of a bubble in a liquid or air as

$$\omega_n^2 = \frac{\gamma(n - 1)n(n + 1)(n + 2)}{a^3\left[(n + 1)\rho_i + n\rho_o\right]} \quad , \qquad (20)$$

and this is identical to the result given by Lamb.[6]

3. LOSSES DUE TO VISCOSITY

As in Morrison et al[5] we shall calculate the Rayleigh dissipation factor, R, to include losses due to viscosity in the equation of motion for the $a_n(t)$. The form given by Landau and Lifshitz[7] for the dissipation factor is

$$R = \frac{1}{2}\,\eta \int (\nabla \mathbf{v}^2) \cdot d\sigma \qquad (21)$$

where η is the viscosity of a particular region and $d\sigma$ is the incremental area directed outward from the volume enclosed by the surface.

We shall consider the region $b < r < r(\theta,t)$ first. For the surface $r = b$, we have

$$d\sigma = -2\pi \hat{r} r^2 d\mu \quad , \tag{22}$$

and since $v_r = 0$ at $r = b$, we have

$$d\sigma \cdot \nabla v^2 = -2\pi r^2 d\mu \, \frac{\partial v_\theta^2}{\partial r} \quad . \tag{23}$$

Using the result of equation (23) in equation (21), we have

$$\int (\nabla v^2) \cdot d\sigma = 8\pi \sum_n \frac{(2n + 1)\dot{a}_n^2}{(n + 1)na^{2n-2}\Gamma_n^2} \quad , \tag{24}$$

for the surface $r = b$. For the surface $r = r(\theta,t)$, we have

$$\int (\nabla v^2) \cdot d\sigma = 8\pi \sum_n \frac{a\dot{a}_n^2}{n\Gamma_n^2} \left[n - 1 - \frac{n(n + 2)}{n + 1} \left(1 - \Gamma_n^2\right)\right] \quad . \tag{25}$$

The total dissipation factor for the region $b < r < r(\theta,t)$ is then given by combining equations (24) and (25) to give

$$R_i = 4\pi\eta_i a \sum_n \left\{\frac{\dot{a}_n^2}{n\Gamma_n^2}\left[n - 1 - \frac{n(n + 2)}{n + 1}\left(1 - \Gamma_n\right)^2\right] + \frac{2(2n + 1)\dot{a}_n^2}{n(n + 1)\Gamma_n^2}\left(1 - \Gamma_{n-1}\right)\right\} . \tag{26}$$

For the exterior region, we have

$$\int (\nabla v^2) \cdot d\sigma = 8\pi a \sum_n \frac{(n + 2)}{n + 1}\dot{a}_n^2 , \tag{27}$$

and the dissipative factor for the exterior region is

$$R_o = 4\pi\eta_o a \sum_n \frac{(n + 2)\dot{a}_n^2}{n + 1} . \tag{28}$$

The total dissipative factor is given by equations (28) and (27), and if this is used in Lagrange's equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{a}_n} - \frac{\partial L}{\partial a_n} = -\frac{\partial R}{\partial \dot{a}_n} \quad , \tag{29}$$

with $L = T - U_s$ and $R = R_i + R_o$, then we have

$$4\pi a^3\left[\frac{\rho_i\left(2n + 1 - n\Gamma_n\right)}{n(n + 1)(2n + 1)\Gamma_n} + \frac{\rho_o}{(n + 1)(2n + 1)}\right]\ddot{a}_n + \frac{4\pi\gamma(n - 1)(n + 2)a_n}{(2n + 1)}$$

$$= -8\pi\eta_i a\left\{\frac{1}{n\Gamma_n^2}\left[n - 1 - \frac{n(n + 2)}{n + 1}\left(1 - \Gamma_n\right)^2\right] + \frac{(2n + 1)(1 - \Gamma_{n-1})}{n(n + 1)\Gamma_n^2}\right\}\dot{a}_n$$

$$- 8\pi\eta\, a\left(\frac{n + 2}{n + 1}\right)\dot{a}_n \quad . \tag{30}$$

If we assume a time dependence of the form $e^{-i\omega t}$ in equation (30), we find that the decay time (imaginary part of ω) is given by

$$\tau_n = \frac{a^2\Gamma_n}{(2n + 1)D_n} \quad , \tag{31}$$

where $$D_n = \eta_i\left\{\left[n^2 - 1 - n(n + 2)\left(1 - \Gamma_n\right)^2\right] + (2n + 1)(1 - \Gamma_{n-1})\right\}$$

$$+ \eta_o\, n(n + 2)\,\Gamma_n^2 \quad .$$

The resonant frequency, ω_n (the real part of ω), is given by

$$\omega_n^2 = \omega_n^2\left(\eta_i = \eta_o = 0\right) - \left(\frac{1}{\tau_n}\right)^2 \quad , \tag{32}$$

for $\frac{1}{\tau_n} < \omega_n\left(\eta_i = \eta_o = 0\right)$ and $\omega_n^2\left(\eta_i = \eta_o = 0\right)$ is given by equation (18).

If in equations (31) and (32) we let $\eta_o = \rho_o = 0$, we recover the result given in Morrison et al,[5] or

$$\tau_n = \frac{\rho_i a^2}{\eta_i (n - 1)}$$

and

$$\omega_n^2 = \frac{\gamma n(n - 1)(n + 2)}{\rho_i a^3} \quad , \tag{33}$$

where the latter result is that of Rayleigh[4] and the decay time is the result of Lamb.[6] Because of the large number of parameters in the expressions for ω_n (equation (32)) and the decay time, τ_n (equation (31)), no general evaluation of the results is possible. Hence, it might be better to use equations (31) and (32) to calculate τ_n and ω_n and compare the results with data taken either in clouds or in a controlled laboratory experiment.

Literature Cited

(1) M. Brook and D. J. Lantham, J. Geophys. Res. 73 (1968), 7137.

(2) C. Musgrove and M. Brook, J. Atmos. Sci. 32 (1975), 2001.

(3) Donald E. Wortman, Possible Use of Two Laser Beams to Determine Particle Size Distribution, Harry Diamond Laboratories, HDL-TR-1878 (January 1979), NTIS #070965.

(4) Lord Rayleigh, Proc. Roy. Soc. 29 (1879) 71.
See also, C. D. Hendricks and J. M. Schneider, Am. J. Phys. 31 (1963), 450.

(5) C. A. Morrison, R. P. Leavitt, and D. E. Wortman, J. Fluid Mech. 104, (1981), 295.

(6) H. Lamb, Hydrodynamics, New York: Dover Publications, (1945), 473.

(7) L. D. Landau and E. M. Lifshitz, Fluid Mechanics New York: Pergamon Press (1959) 54.